

The maximum size of a partial spread in $H(5, q^2)$ is $q^3 + 1$

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Abstract

In this paper, we show that the largest maximal partial spreads of the hermitian variety $H(5, q^2)$ consist of $q^3 + 1$ generators. Previously, it was only known that q^4 is an upper bound for the size of these partial spreads. We also show for $q \geq 7$ that every maximal partial spread of $H(5, q^2)$ contains at least $2q + 3$ planes. Previously, only the lower bound $q + 1$ was known.

Keywords: ovoid, polar space, Hermitian variety

1 Introduction

A *spread* of a polar space is a set of mutually skew generators that partition the space. In the case of finite hermitian polar spaces $H(d, q^2)$, a spread consists of $q^d + 1$ generators. J. Thas [9] showed that the finite hermitian variety $H(2n + 1, q^2)$ over the field F_{q^2} possesses no spread. Recently, D. Luyckx was able to considerably improve this result in the case $n = 2$. She showed in [5] that every partial spread (set of mutually skew generators) of $H(5, q^2)$ consists of at most q^4 planes. She also constructed examples for partial spreads of $H(4n + 1, q^2)$, $n \geq 1$, of size $q^{n+1} + 1$.

For $n = 2$ the construction is as follows. By choosing a suitable hermitian form to define $H(5, q^2)$ in $\text{PG}(5, q^2)$, the restriction to field elements in F_q produced a symplectic polar space $W(5, q)$ in a Baer-subgeometry. It is known that this symplectic polar space has spreads, which have size $q^3 + 1$. These $q^3 + 1$ planes form a partial spread in $H(5, q^2)$. The main result of the present paper is to prove that this construction is optimal.

Theorem 1.1 *Every partial spread of $H(5, q^2)$ has at most $q^3 + 1$ elements.*

We checked for $q = 2$ that $H(5, q^2)$ has partial spreads of cardinality $q^3 + 1$ that do not arise from a symplectic spread as above. D. Luyckx in her paper also shows that a maximal partial spread of $H(2n + 1, q^2)$ must have size at least $q + 1$. It is likely that this bound is far away from the reality, but we can only make a slight improvement.

Theorem 1.2 *A maximal partial spread of $H(5, q^2)$ has at least $2q + 3$, if $q \geq 7$, at least $2q + 2$ generators for $q \in \{3, 4, 5\}$ and at least $2q + 1 = 5$ generators for $q = 2$.*

2 The proof

Consider a partial spread \mathcal{S} of the Hermitian variety $H(5, q^2)$ embedded in $\text{PG}(5, q^2)$. The points that are covered by the planes of \mathcal{S} will be called covered points. The planes contained in $H(5, q^2)$ are called hermitian planes. Since the partial spread is maximal, every hermitian plane contains a covered point. On the other hand, a hermitian plane that is not in the partial spread can meet at most one of the planes of \mathcal{S} in a line. The hermitian planes that are not in \mathcal{S} and do not contain a line of a plane of \mathcal{S} will be called *free planes*. Finally we put $x := q^4 + 1 - |\mathcal{S}|$.

Lemma 2.1 *Every covered point lies on x free planes. Every uncovered points of $H(5, q^2)$ lies on $q^3 + q + x$ free planes.*

Proof. Let P be an uncovered point. For every plane $\pi \in \mathcal{S}$ the subspace $\langle P, P^\perp \cap \pi \rangle$ is a hermitian plane on P meeting π in a line. Hence P lies on exactly $|\mathcal{S}|$ hermitian planes that meet a plane of \mathcal{S} in a line. Then the number of free planes on P is $(q + 1)(q^3 + 1) - |\mathcal{S}| = q^3 + q + x$. Now consider a covered point P in a plane π_0 of \mathcal{S} . The other planes π of \mathcal{S} still give rise to the planes $\langle P, P^\perp \cap \pi \rangle$, but there are $(q^2 + 1)q$ hermitian planes on P that meet π_0 in a line, so now the number of free planes on P is $q^3 + q$ smaller than for the uncovered points. \square

This lemma shows that $x \geq 0$ and hence $|\mathcal{S}| \leq q^4 + 1$. This was noticed by D. Luyckx in [5]. The lemma has another interesting consequence. Consider

the multiset \mathcal{M} consisting of the free planes and $q^3 + q$ copies of each plane of \mathcal{S} . Then every hermitian point is covered exactly $q^3 + q + x$ times by planes of this multiset. This has powerful consequences. In order to prove these, we need the following remarkable property of hermitian varieties noticed by Thas [9].

Result 2.2 *Let π_1, π_2 and π be three distinct generators of $H(2n + 1, q^2)$. Then the points of π that lie on a line of $H(2n + 1, q^2)$ meeting π_1 and π_2 form a hermitian variety $H(n, q^2)$ in π .*

In the degenerate situation $n = 1$, we mean by a hermitian variety $H(1, q^2)$ a set of $q + 1$ collinear points. We remark that this property can be verified easily in the case $n = 1$ by using the duality of $H(3, q^2)$ and $Q^-(5, q)$.

Lemma 2.3 *For two different planes π_1, π_2 of \mathcal{S} the number of free planes intersecting both is equal to*

$$y := x(q^3 + 1) - (q^3 + q)(q^2 - q + 1)(q - 1).$$

Proof Let π_1 and π_2 be two different planes of \mathcal{S} . Then the union U of all hermitian lines meeting π_1 and π_2 has size $(q^4 + q^2 + 1)(q^4 + 1)$. Now consider the multiset \mathcal{M} constructed above whose planes cover every hermitian point $q^3 + q + x$ times. We count incident pairs $(P, \pi) \in U \times \mathcal{M}$. Each point of U occurs in $q^3 + q + x$ pairs.

The $q^3 + q$ copies of π_1 and π_2 in \mathcal{M} occur each in $q^4 + q^2 + 1$ pairs. A plane of \mathcal{M} that is skew to π_1 and π_2 occurs $q^3 + 1$ times by the above result; this applies to the $(|S| - 2)(q^3 + q)$ of planes of $\mathcal{S} \setminus \{\pi_1, \pi_2\}$. For the free planes in \mathcal{M} there are three possibilities. They can be skew to π_1 and π_2 . Then they also meet U in $q^3 + 1$ points. They can meet π_1 and π_2 in one point. Then they meet U in a line, so these free planes occur in $q^2 + 1$ pairs. We denote by y the number of such free planes. Then the number of free planes that meet exactly one of π_1 and π_2 is $2(q^4 + q^2 + 1)x - 2y$ by Lemma 2.1. It follows from Result 2.2 that these free planes occur in $1 + (q + 1)q^2$ pairs.

Thus, each plane of \mathcal{M} occurs in $q^3 + 1$ pairs, except that $2(q^3 + q)$ occur $q^4 - q^3 + q^2$ extra times, $2(q^4 + q^2 + 1)x - 2y$ occur q^2 extra times, and y occur $q^3 + 1 - (q^2 + 1) = q^3 - q^2$ times less. Hence

$$\begin{aligned} |U|(q^3 + q + x) &= |\mathcal{M}|(q^3 + 1) + 2(q^3 + q)(q^4 - q^3 + q^2) \\ &\quad + [2(q^4 + q^2 + 1)x - 2y]q^2 - y(q^3 - q^2) \end{aligned}$$

As the planes of \mathcal{M} cover $H(5, q^2)$ exactly $q^3 + q + x$ times, we have $|\mathcal{M}| = (q^5 + 1)(q^3 + q + x)$. Simplifying gives y as stated. \square

We have $|\mathcal{F}| = |\mathcal{M}| - |\mathcal{S}|(q^3 + q)$. Using the size for $|\mathcal{M}|$ from the above proof, we find

$$\sum_{F \in \mathcal{F}} 1 = |\mathcal{F}| = (q^5 - q^4)(q^3 + q) + x(q^5 + q^3 + q + 1)$$

For $F \in \mathcal{F}$ denote by c_F the number of points of F that are covered by planes of the partial spread \mathcal{S} . Counting incident pairs (P, F) with points covered by \mathcal{S} and free planes F , Lemma 2.1 gives

$$\sum_{F \in \mathcal{F}} c_F = |\mathcal{S}|(q^4 + q^2 + 1)x.$$

Counting triples (P, P', F) of different points covered by \mathcal{S} and free planes F with $P, P' \in F$, the preceding lemma gives

$$\sum_{F \in \mathcal{F}} c_F(c_F - 1) = |\mathcal{S}|(|\mathcal{S}| - 1)y.$$

Using these three equalities to evaluate the Cauchy-Schwarz-inequality

$$|\mathcal{F}| \sum_{F \in \mathcal{F}} c_F^2 \geq \left(\sum_{F \in \mathcal{F}} c_F \right)^2$$

using $x = q^4 + 1 - |\mathcal{S}|$ and $s := |\mathcal{S}|$, gives

$$\begin{aligned} 0 \leq & sq(q^2 - q + 1)(q^3 + 1 - s)(q^{11} + q^{10} + q^9 - sq^7 + q^7 + 2q^6 - 2sq^6 \\ & - sq^4 + q^4 - sq^3 + s^2q^3 - sq^2 + q^2 + s^2q - 2sq + q - s^2 + 2s - 1). \end{aligned}$$

It follows that $|\mathcal{S}| \leq q^3 + 1$. Here we used that we have $|\mathcal{S}| \leq q^4 + 1$, see above.

Now suppose that $|\mathcal{S}| = q^3 + 1$. Then we have equality and this implies that all planes of \mathcal{F} have the same number f of covered points. The above equations for $\sum c_F$ and $|\mathcal{F}|$ show that this number is $q^2 - q + 1$. We also have $|\mathcal{F}| = q^6(q^3 - 1)$ and the number y of planes of \mathcal{F} meeting two planes of \mathcal{F} is

$$y = (q^4 + q^2 + 1)(q - 1)^2q.$$

This information shows that all spreads of size $q^3 + 1$ behave similar. However, we also mention that there might exist different spreads.

3 Small maximal partial spreads

In order to prove a lower bound for small maximal partial spreads of $H(5, q^2)$, we need to calculate some numbers. The crucial point of our counting argument is that the number of planes of $H(5, q^2)$ that meet three mutually skew planes of $H(5, q^2)$ is independent of the three planes chosen.

Lemma 3.1 (a) *Every plane of $H(5, q^2)$ meets $(q^4 + q^2 + 1)(q^4 + q)$ other planes of $H(5, q^2)$.*

(b) *If π_1 and π_2 are mutually skew planes of $H(5, q^2)$, then there exist exactly $(q^4 + q^2 + 1)(q^3 - q^2 + q + 1)$ planes of $H(5, q^2)$ meet π_1 and π_2 .*

(b) *If π_1, π_2 and π are three mutually skew planes of $H(5, q^2)$, then $q^6 - 2q^5 + 3q^4 + q + 1$ planes of $H(5, q^2)$ meet π_1, π_2 and π .*

Proof (a) Each of the $q^4 + q^2 + 1$ lines of a plane π of $H(5, q^2)$ lies in q further planes. A point of π lies in $(q + 1)(q^3 + 1)$ planes of $H(5, q^2)$, of which one is π and $(q^2 + 1)q$ other ones meet π in a line, so q^4 of which meet π only in this point. Thus there exist $(q^4 + q^2 + 1)q^4$ planes in $H(5, q^2)$ that meet π in a unique point.

(b) Consider a point $P \in \pi_1$. The number of planes on P that meet π_2 can be counted in the quotient geometry on P : Given two skew lines l_1 and l_2 in $H(3, q^2)$, there are exactly $1 + (q^2 + 1)q$ lines that meet l_2 and exactly $q^2 + 1$ of these meet also l_1 . Thus, P lies in $q^3 + q + 1$ planes of $H(5, q^2)$ that meet π_2 and exactly $q^2 + 1$ of these meet π_1 in a line. It follows that there exists $q^4 + q^2 + 1$ planes of $H(5, q^2)$ that meet π_1 in a line and π_2 in a point, and there are $(q^4 + q^2 + 1)(q^3 - q^2 + q)$ planes in $H(5, q^2)$ that meet π_1 in a unique point and that meet also π_2 .

(c) First we recall from Result 2.2 that we find a hermitian curve $H = H(2, q^2)$ in the plane π consisting of those points of π that lie on a line of $H(5, q^2)$ that meets π_1 and π_2 . Alternatively, one can say that H consists of the points $P \in \pi$ such that the planes $\langle P, P^\perp \cap \pi_1 \rangle$ and $\langle P, P^\perp \cap \pi_2 \rangle$ meet in a line l (and this is the line of $H(5, q^2)$ on P that meets π_1 and π_2).

Consider a point $P \in \pi$. Then every plane of $H(5, q^2)$ on P that meets π_1 and π_2 , meets π_i in a point of the plane $E_i := \langle P, P^\perp \cap \pi_i \rangle$. Going into the quotient space P^\perp/P , in which we see a $H(3, q^2)$, the planes E_1, E_2 and π

become lines l_1, l_2, l . The number of planes of $H(5, q^2)$ on P that meet also π_1 and π_2 is equal to the number of lines in the $H(3, q^2)$ that meet l_1 and l_2 . If $P \in H$, then the lines l_1 and l_2 meet in a point and l is disjoint to l_1 and l_2 . In this case there are $q + 1$ lines meeting l_1 and l_2 and one of these meets also l . If $P \notin H$, then l_1, l_2 and l are mutually skew, so there are $q^2 + 1$ lines in $H(3, q^2)$ that meet l_1 and l_2 and, by Result 2.2, exactly $q + 1$ of these meet also l .

Thus the number of planes of $H(5, q^2)$ on P that meet π_1 and π_2 is $q + 1$ in the first case and $q^2 + 1$ in the second case. Also in the first case one and in the second case $q + 1$ of these planes meet π in line. Thus, from the planes of $H(5, q^2)$ that meet π_1 and π_2 , exactly

$$(q^3 + 1)q + (q^4 - q^3 + q^2)(q^2 - q) = q^6 - 2q^5 + 3q^4 - q^3 + q$$

meet π in a unique point, and

$$\frac{(q^3 + 1) \cdot 1 + (q^4 - q^3 + q^2)(q + 1)}{q^2 + 1} = q^3 + 1$$

meet π in a line. \square

Remark. In the previous proof, one can also show the following. A tangent line of the hermitian curve H in π lies on a unique plane of $H(5, q^2)$ meeting π_1 and π_2 , whereas the other lines of π do not lie in planes that meet π_1 and π_2 . This explains the term $q^3 + 1$ for the number of planes meeting π , π_1 and π_2 .

To obtain from this information a lower bound we use a standard counting technique, see for example [7]. Suppose that \mathcal{F} is a maximal partial spread of $H(5, q^2)$. Let n_i be the number of planes of $H(5, q^2)$ that are not in \mathcal{F} and that meet exactly i planes of \mathcal{F} . Then $n_0 = 0$ as the spread is maximal. Also

$$\sum_{i \geq 1} n_i = (q + 1)(q^3 + 1)(q^5 + 1) - |\mathcal{F}| \quad (1)$$

$$\sum_{i \geq 1} n_i i = |\mathcal{F}|(q^4 + q^2 + 1)(q^4 + q) \quad (2)$$

$$\sum_{i \geq 1} n_i i(i - 1) = |\mathcal{F}|(|\mathcal{F}| - 1)(q^4 + q^2 + 1)(q^3 - q^2 + q + 1) \quad (3)$$

$$\sum_{i \geq 1} n_i i(i-1)(i-2) = |\mathcal{F}|(|\mathcal{F}|-1)(|\mathcal{F}|-2)(q^6 - 2q^5 + 3q^4 + q + 1). \quad (4)$$

The first equation holds, since $H(5, q^2)$ has $(q+1)(q^3+1)(q^5+1)$ generators. The second equation follows from a double counting argument, since each generator meets $(q^4 + q^2 + 1)(q^4 + q)$ other generators. Finally the third and forth equation follow from the lemma by counting suitable triple and 4-tuples. These equation enable us to calculate the sum

$$S := \sum_{i \geq 1} n_i(i-1)(i-3)(i-4).$$

Clearly $S \geq 0$. Simplifying using the above equation yields (here we put $|\mathcal{F}| = 2q + 2 + x$)

$$\begin{aligned} 0 \leq & (q^6 - 2q^5 + 3q^4 + q + 1)x^3 \\ & + (q^7 + 6q^2 + 9q^4 + 4q + 2q^5 - 10q^3 - 2 - 4q^6)x^2 \\ & + (-22q^4 - 1 - 7q^7 + 4q^2 + q^6 - 6q^3 + 14q^5 - 9q + 4q^8)x + 2 \\ & - 2q^8 - 16q^4 + 14q^6 - 12q^5 - 10q^2 - 8q^3 + 8q^7. \end{aligned}$$

For large q we immediately see that $x > 0$ and hence $|\mathcal{F}| \geq 2q + 3$. In fact, this holds for $q \geq 7$. For $q = 5$ and $q = 3$ we still deduce $x > -1$, that is $|\mathcal{F}| \geq 2q + 2$, and for $q = 2$ we find $|\mathcal{F}| \geq 2q + 1 = 5$.

We remark that the same technique can be applied to $H(2n+1, q^2)$ for any n , only one has to calculate the numbers as in Lemma 3.1. For example, if $n = 1$, the numbers are easy to calculate, where again for the last number Result 2.2 is needed. Thus, if \mathcal{F} is a partial spread of $H(3, q^2)$, then

$$\begin{aligned} \sum_{i \geq 1} n_i &= (q+1)(q^3+1) - |\mathcal{F}| \\ \sum_{i \geq 1} n_i i &= |\mathcal{F}|(q^2+1)q \\ \sum_{i \geq 1} n_i i(i-1) &= |\mathcal{F}|(|\mathcal{F}|-1)(q^2+1) \\ \sum_{i \geq 1} n_i i(i-1)(i-2) &= |\mathcal{F}|(|\mathcal{F}|-1)(|\mathcal{F}|-2)(q+1). \end{aligned}$$

The calculating the same sum as above gives $|\mathcal{F}| \geq 2q + 1$ for $q = 2, 3$, and $|\mathcal{F}| \geq 2q + 2$ for $q \geq 4$

4 Particular results for small values of q

Using the computer algebra system GAP [6] and the package pg [3], we constructed all maximal partial spreads of $H(5, 4)$ and $H(3, q^2)$, $q = 2, 3$. For $H(5, 4)$ we were interested in the different kind of maximal partial spreads that exist, and some of their geometric properties. For $H(3, q^2)$, $q = 2, 3$ we were more interested in the possible sizes that maximal partial spreads can have.

maximal partial spreads of $H(5, 4)$

The following table summarizes the information on the different maximal partial spreads of $H(5, 4)$.

class	size	stabilizer group	order of the group
symplectic	9	$L_2(8) : C_3$	$9 \cdot 8 \cdot 7 \cdot 3 = 1512$
derivation 1	9	$C_3 \wr S_3$	$3^3 \cdot 2 \cdot 3 = 162$
derivation 2	9	$(C_2^3 : C_7) : C_3$	$2^3 \cdot 7 \cdot 3 = 168$
derivation 3	7	C_2	2
derivation 4	7	$D_{10} \cong C_5 : C_2$	10

There are, up to collineation, 3 examples of maximal partial spreads of size 9 and two examples of maximal partial spreads of size 7. The (up to collineation unique) spread of $W(5, 2)$ (embedded in $H(5, 4)$) is called the symplectic example. Its stabilizer group, i.e. the subgroup of $\text{PGU}(6, 4)$ stabilizing the spread planewise, has actually order 9072, but its action on the planes of the maximal partial spread S is isomorphic with $L_2(8) : C_3$, which is a group of order 1512. The reason is that a collineation of $W(5, 2)$ embedded in $H(5, 4)$ can be extended in several ways to a collineation of $H(5, 4)$. The group acts 3-transitively on the planes, the normal subgroup $L_2(8)$ acts sharply 3-transitively on the planes and is simple.

Suppose that S is a maximal partial spread of size 9. Consider a triple T of planes of S . Denote with $F(T)$ the set of free planes of $H(5, 4)$ intersecting every plane of T . Suppose that S is the symplectic example. Then for any triple T the set $F(T)$ contains at least one triple of mutually skew planes intersecting no other planes of S than the three planes of the chosen triple.

We say that the set $F(T)$ satisfies condition D. Hence replacing the planes from the triple with such a triple of mutually skew planes yield a new spread of size 9. We call this procedure *derivation*. Since the stabilizer group of S acts 3-transitively on the planes of S , it is clear that a derivation from any chosen triple yields the same spread. We call this example a derivation 1 example. Its stabilizer group has order 162 and acts transitively on the planes of the spread. The action is imprimitive. A non-trivial block system exists where each block contains exactly three planes, the action of the group on the blocks is isomorphic to S_3 (the symmetric group on three elements). The subgroup stabilizing each block is elementary abelian and isomorphic with C_3^3 . The derivation process can be executed on this example, but not all chosen triples will yield the same spread now. For some triples T , the set $F(T)$ will even not satisfy condition D, but if it does, the derivation can be a symplectic example, a derivation 1 or a derivation 2 example.

Suppose that S is a derivation 2 example. Its stabilizer group fixes exactly one plane, the action of the group is 2-transitive on the 8 remaining planes and is isomorphic to $(C_2^3 : C_7) : C_3$. The normal subgroup $C_2^3 : C_7$ acts sharply 2-transitive on the 8 remaining planes. Again the process of derivation can be executed and if, for a chosen triple T , the set $F(T)$ satisfies condition D, then the derivation of S is always a derivation 1 example.

Suppose now that S is a derivation 1 or a derivation 2 example. It is always possible to find a triple T of planes of S such that $F(T)$ does not satisfy condition D, such that $F(T)$ contains a plane π that intersects two planes of $S \setminus T$, such that $F(T)$ contains two more planes π' and π'' , not intersecting the planes of $S \setminus T$ and such that π , π' and π'' are mutually skew. Removing the two planes of $S \setminus T$ that intersect π , and the three planes of T , and adding the three planes π , π' and π'' , yields a maximal partial spread of size 7. Starting from a derivation 1 or derivation 2 example, we can always construct derivation 3 and derivation 4 examples if we choose a suitable triple T .

If S is a derivation 3 example, then its stabilizer group fixes one plane of S . Its action on the remaining six planes is involutory, the stabilizer group is isomorphic with C_2 . If S is a derivation 4 example, then its stabilizer group has order 10, fixes no plane, but does not act transitively on the planes of S . There are two orbits, one has length 7, the second one has length 2. The stabilizer group is isomorphic with $D_{10} \cong C_5 : C_2$, the dihedral group of

order 10.

We recall that in the case $H(5, 4)$ the theoretic lower bound from the previous section was 5. We see that in reality the smallest maximal partial spread has size 7.

Maximal partial spreads of $H(3, q^2)$, $q = 2, 3$

From the previous section we know that a maximal partial spread of $H(3, q^2)$, $q = 2, 3$ contains at least $2q + 1$ planes. That $H(3, q^2)$ has a maximal partial spread of size $q^2 + 1$ is observed in [4], as in the $H(5, q^2)$, we can embed the symplectic polar space (now of rank 2) into $H(3, q^2)$. Alternatively, one says that $W(3, q)$ is a subquadrangle of the generalized quadrangle $H(3, q^2)$ [8]. It is known that $W(3, q)$ has spreads. Suppose that S is a spread of $W(3, q)$, then one can show that the extension in $H(3, q^2)$ is a maximal partial spread as follows. Consider the dual situation, i.e. we interchange the role of the points and the lines in the generalized quadrangles $W(3, q)$ and $H(3, q^2)$. The dual of $W(3, q)$ is isomorphic with $Q(4, q)$ and the dual of $H(3, q^2)$ is isomorphic with $Q^-(5, q)$. Hence in the dual situation we consider an *ovoid* of $Q(4, q)$ embedded in $Q^-(5, q)$. An ovoid of $Q(4, q)$ is a set \mathcal{O} of points of $Q(4, q)$ such that every line of $Q(4, q)$ meets the ovoid in exactly one point. (remark that this is exactly the dual of a spread of $W(3, q)$). Considering the ovoid \mathcal{O} of $Q(4, q)$, we have to show that any point of $Q^-(5, q)$ is collinear on $Q^-(5, q)$ with at least one point of \mathcal{O} . Consider a point $p \in Q^-(5, q) \setminus Q(4, q)$, then all points collinear with p lie in a hyperplane π_4 of $\text{PG}(5, q)$. Consider the 4-dimensional space π_4 containing $Q(4, q)$. Then π intersects π_4 in a 3-dimensional space, and it is known that each 3-dimensional space contains exactly $1 \bmod p$ points of \mathcal{O} , with $q = p^h$ [1, 2]. This shows that any spread of $W(3, q)$ constitutes a maximal partial spread of $H(3, q^2)$, of size $q^2 + 1$.

Using an exhaustive search, we found that $H(3, q^2)$, $q = 2$ has, up to collineation, 1 maximal partial spread of size 5 and 1 maximal partial spread of size 6. Hence the lower bound $2q + 1$ is reached in this case. For $q = 3$, we found that $H(3, q^2)$ has maximal partial spreads of size 10, 11, 12, 13 and 16. In this case the lower bound $2q + 1$ is not reached. We observe that the size of the symplectic spread ($q^2 + 1$) is the real lower bound for $q = 2, 3$, so the question arises whether this is true for general q . We remark that more results on maximal partial spreads of $H(3, q^2)$, $q = 2, 3$ can be found in [4].

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